A SOLUTION TO A PROBLEM OF CASSELS AND DIOPHANTINE PROPERTIES OF CUBIC NUMBERS

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Abstract. We prove that almost any pair of real numbers α, β, satisfies the following inhomogeneous uniform version of Littlewood's conjecture:

$$\forall \gamma, \delta \in \mathbb{R}, \quad \liminf_{|n|\to\infty} |n| \langle n\alpha - \gamma\rangle\langle n\beta - \delta\rangle = 0, \tag{0.1}$$

where $\langle\cdot\rangle$ denotes the distance from the nearest integer. The existence of even a single pair that satisfies (0.1), solves a problem of Cassels [Ca] from the 50's. We then prove that if $1, \alpha, \beta$ span a totally real cubic number field, then α, β, satisfy (0.1). This generalizes a result of Cassels and Swinnerton-Dyer, which says that such pairs satisfies Littlewood's conjecture. It is further shown that if α, β are any two real numbers, such that $1, \alpha, \beta$, are linearly dependent over $\mathbb{Q}$, they cannot satisfy (0.1). The results are then applied to give examples of irregular orbit closures of the diagonal groups of a new type. The results are derived from rigidity results concerning hyperbolic actions of higher rank commutative groups on homogeneous spaces.

1. Introduction

1.1. **Notation.** We first fix our notation and define the basic objects to be discussed in this paper. Let X_d denote the space of d-dimensional unimodular lattices in $\mathbb{R}^d$ and let Y_d denote the space of translates of such lattices. Points of Y_d will be referred to as *grids*, hence for $x \in X_d$ and $v \in \mathbb{R}^d$, $y = x + v \in Y_d$ is the grid obtained by translating the lattice x by the vector v. We denote by π the natural projection

$$Y_d \xrightarrow{\pi} X_d, \quad x + v \mapsto x. \tag{1.1}$$

For each $x \in X_d$, we identify the fiber $\pi^{-1}(x)$ in Y_d with the torus $\mathbb{R}_d/x$. Let $N : \mathbb{R}^d \to \mathbb{R}$ denote the function $N(w) = \prod_1^d w_i$. For a grid $y \in Y_d$, we define the *product set* of y to be

$$P(y) = \{N(w) : w \in y\} \tag{1.2}$$

In this paper we shall study properties of the product set. We will mainly be interested in density properties and the values near zero. We denote

$$N(y) = \inf \{|N(w)| : w \in y\}. \tag{1.3}$$

The ambiguous use of the symbol N both for a function on $\mathbb{R}^d$ and for a function on Y_d and the lack of appearance of the dimension d in the notation should not cause any

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confusion. The *inhomogeneous minimum* of a lattice $x \in X_d$ is defined by

$$\mu(x) = \sup\left\{N(y) : y \in \pi^{-1}(x)\right\}. \tag{1.4}$$

The *inhomogeneous Markov spectrum* (or just the spectrum) is defined by

$$\mathcal{S}_d = \{\mu(x) : x \in X_d\}. \tag{1.5}$$

A more geometric way to visualize the above notions is the following: The *star body* of radius $\epsilon > 0$ is the set $S_\epsilon = \left\{w \in \mathbb{R}^d : |N(w)| < \epsilon\right\}$. In terms of star bodies, for a grid $y \in Y_d$, $N(y) = \inf\{\epsilon : S_\epsilon \cap y \neq \emptyset\}$ and for a lattice $x \in X_d$, $\mu(x)$ is the least number such that for any $\epsilon > \mu(x)$, the star body S_ϵ intersects all the grids of x or equivalently, S_ϵ projects onto the torus $\pi^{-1}(x)$ under the natural projection.

1.2. **Dimension 2.** In [D] Davenport showed (generalizing a result of Khintchine) that for any $x \in X_2$ one has $\mu(x) > \frac{1}{128}$, hence the spectrum $\mathcal{S}_2$ is bounded away from zero. The constant $\frac{1}{128}$ is not optimal and much work has been done to improve Davenport's lower bound of the spectrum (see [Ca],[Ca2],[D] and the references therein). The set $\{y \in \pi^{-1}(x) : \mu(y) > 0\}$ (where $x \in X_2$ is arbitrary) has been investigated too and in a recent work [ET] it was shown that it has full Hausdorff dimension and in fact that it is a winning set for Schmidt's game.

1.3. **Cassels problem.** In his book [Ca] (p. 307), Cassels raised the following natural question:

Problem 1.1 (Cassels). *In dimension $d \geq 3$, is the infemum of the spectrum $\mathcal{S}_d$ equal to zero.*

We answer Cassels' problem affirmatively. In fact we show that the infemum is attained and give explicit constructions of lattices attaining the minimum. The following theorem is a consequence of corollary 4.8(1):

Theorem 1.2. *For $d \geq 3$, almost any lattice $x \in X_d$ (with respect to the natural probability measure) satisfies $\mu(x) = 0$.*

1.4. **Diophantine approximations.** Of particular interest to Diophantine approximations, are lattices of the following forms: Let $v \in \mathbb{R}^{d-1}$ be a column vector. Denote

$$h_v = \begin{pmatrix} I_{d-1} & v \\ 0 & 1 \end{pmatrix}, \quad g_v = \begin{pmatrix} 1 & v^t \\ 0 & I_{d-1} \end{pmatrix} \tag{1.6}$$

where I_{d-1} denotes the identity matrix of dimension $d-1$ and the 0's denote the corresponding trivial vectors. Let $x_v, z_v \in X_d$, denote the lattices spanned by the columns of h_v and g_v respectively. For $\gamma \in \mathbb{R}$, denoting by $\langle\gamma\rangle$, the distance from γ to the nearest integer, an easy calculation shows that the statements

$$\forall \vec{\gamma} \in \mathbb{R}^{d-1} \quad \liminf_{|n| \to \infty} |n| \prod_{1}^{d-1} \langle n v_i - \gamma_i \rangle = 0, \tag{1.7}$$

$$\forall\gamma\in\mathbb{R}\quad \liminf_{\prod|n_i|\to\infty}\prod_1^{d-1}|n_i|\,\langle\sum_1^{d-1}n_iv_i-\gamma\rangle=0, \tag{1.8}$$

imply that $\mu(x_v)=0$ and $\mu(z_v)=0$ respectively.

Theorem 1.3. *Let $d\geq 3$*

1. *For almost any $v\in\mathbb{R}^{d-1}$ (with respect to Lebesgue measure)* (1.7) *and* (1.8) *are satisfied and in particular $\mu(x_v)=\mu(z_v)=0$.*
2. *Nonetheless, if $\dim span_{\mathbb{Q}}\{1,v_1,\ldots,v_{d-1}\}\leq 2$ then $\mu(x_v),\mu(z_v)$ are positive.*

Part (1) of the above theorem is a consequence of corollary 4.8. Part (2) follows from known results in dimension 2 and will be proved in §6.
Perhaps the most interesting amongst the results in this paper is the following theorem which shows that certain pairs of algebraic numbers are generic. The proof follows from corollary 5.2 and corollary 4.8(3).

Theorem 1.4. *If $1,\alpha,\beta$ form a basis for a totally real cubic number field, then*

$$\forall\gamma,\delta\in\mathbb{R}\quad \liminf_{|n|\to\infty}|n|\langle n\alpha-\gamma\rangle\langle n\beta-\delta\rangle=0, \tag{1.9}$$

$$\forall\gamma\in\mathbb{R}\quad \liminf_{|nm|\to\infty}|nm|\langle n\alpha+m\beta-\gamma\rangle=0. \tag{1.10}$$

1.5. **Remarks.**

1. Cassels and Swinnerton-Dyer have shown [CaSD] that any real pair α,β, belonging to the same cubic totally real field, satisfies Littlewood's conjecture, i.e. satisfies (1.9) with $\gamma=\delta=0$. Thus theorem 1.4 together with theorem 1.3(2), can be viewed as a strengthening of their result.
2. As Cassels points out in his book [Ca], problem 1.1 belongs to a family of problems for various forms (other than N). Barnes [Ba] solved an analogous problem with N replaced by an indefinite quadratic form in $d\geq 3$ variables. Our method, when adapted appropriately, seem to give a different proof of Barnes' result.
3. In a recent paper [Bu], Y. Bugeaud raised (independently of Cassels) the question of existence of pairs $\alpha,\beta\in\mathbb{R}$ which satisfy (0.1).
4. Our methods are dynamical and rely on rigidity results such as Ratner's theorem [R], the results and techniques appearing in [LW] and the extension of Furstenberg's times 2 times 3 theorem [F] due to Berend [B]. But, although the usual ergodic theoretic arguments provide existence only, our results provide us with concrete examples of numbers and lattices with nontrivial dynamical and Diophantine properties.

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2. Basic notions, groups and homogeneous spaces

When $d \geq 2$ is fixed we denote $G = SL_d(\mathbb{R}) \ltimes \mathbb{R}^d$, $G_0 = SL_d(\mathbb{R})$ and $V = \mathbb{R}^d$. We shall identify G_0, V with the corresponding subgroups of G. Denote by $A < G_0$ the subgroup of diagonal matrices with positive diagonal entries. The Lie algebra of A is identified with the Euclidean $d-1$ dimensional space

$$\mathfrak{a} = \left\{ \mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d : \sum_1^d t_i = 0 \right\}. \tag{2.1}$$

A is isomorphic to the additive group $\mathfrak{a}$, via the exponent map $\exp : \mathfrak{a} \to A$ given by $\exp(\mathbf{t}) = diag(e^{t_1}, \ldots, e^{t_d})$. We denote the inverse of exp by log. The *roots* of A are the linear functionals on $\mathfrak{a}$ of the following forms

$$\forall 1 \leq i \neq j \leq d, \quad \mathbf{t} \mapsto t_i - t_j; \quad \forall 1 \leq k \leq d, \quad \mathbf{t} \mapsto t_k. \tag{2.2}$$

The set of roots will be denoted by Φ. As suggested in (2.2), we say that a root $\alpha \in \Phi$ corresponds to a pair $1 \leq i \neq j \leq d$ or to an index $1 \leq k \leq d$. To each root $\alpha \in \Phi$, there corresponds a one parameter unipotent subgroup $\{u_\alpha(t)\}_{t \in \mathbb{R}} < G$, called the *root group*, for which the following equation is satisfied

$$a u_\alpha(t) a^{-1} = u_\alpha(e^{\alpha(\log(a))} t). \tag{2.3}$$

When the root α corresponds to a pair $i \neq j$, we sometime denote $u_\alpha(t) = u_{ij}(t)$. In this case $u_{ij}(t) \in G_0$ is the matrix all of whose entries are zero, except for the ij'th which is equal to t and the diagonal entries which are equal to 1. When α corresponds to $1 \leq k \leq d$, we sometime denote $u_\alpha(t) = u_k(t)$. In this case, $u_k(t) \in V$ is the vector, te_k, where e_k is the k'th standard vector. We sometime abuse notation and write, for a root $\alpha \in \Phi$ and $a \in A$, $\alpha(a)$ instead of $\alpha(\log(a))$.

For an element $a \in A$ we define the *stable horospherical subgroup* of G corresponding to it to be $U^-(a) = \{(g, v) \in G : a^n (g, v) a^{-n} \to_{n \to \infty} e\}$, and the *unstable horospherical subgroup* to be $U^+(a) = U^-(a^{-1})$. An element $b \in A$ is called *regular* if for any root $\alpha \in \Phi$, $\alpha(b) \neq 0$. For $b \in A$, any element $g \in G$ which is close enough to e, has a unique decomposition $g = cu^+u^-$, where c centralizes b, $u^+ \in U^+(b), u^- \in U^-(b)$ and c, u^+, u^- lie in corresponding neighborhoods of e. If b is regular then the centralizer of b is A.
The linear action of G_0 on $\mathbb{R}^d$ induces a transitive action of G_0 on X_d. The stabilizer of the lattice $\mathbb{Z}^d \in X_d$ is $\Gamma_0 = SL_d(\mathbb{Z})$. This enables us to identify X_d with the homogeneous space G_0/Γ_0. For $g \in G_0$, we denote $\bar{g} = g\Gamma_0$. $\bar{g} \in X_d$ represents the lattice spanned by the columns of the matrix g. In a similar manner we identify Y_d with G/Γ, where $\Gamma = SL_d(\mathbb{Z}) \ltimes \mathbb{Z}^d$. For $(g, v) \in G$, $(g, v)\Gamma$ represents the grid $\bar{g} + v$. Γ (resp Γ_0) is a lattice in G (resp G_0). The G (resp G_0) invariant probability measure on Y_d (resp X_d) will be referred to as the *Haar measure*. G_0 and its subgroups act on X_d, Y_d and the action commutes with the projection $\pi : Y_d \to X_d$. Finally, we say that a grid $y = x + v$ is *rational*, if v belongs to the $\mathbb{Q}$-span of the lattice x. This is equivalent to saying that $y \in \pi^{-1}(x)$ is a torsion element.

3. Compact A orbits

The following classification theorem, essentially goes back to [Bac]. A modern proof can be found in [LW] or [Mc]. Before stating it, let us recall some notions from number theory. A *totally real number field* is a finite extension of $\mathbb{Q}$, all of whose embeddings into $\mathbb{C}$ are real. A *lattice* in a number field, is the $\mathbb{Z}$-span of a basis of the field over $\mathbb{Q}$. Let K be a totally real number field of degree d and let $\sigma_i, i = 1 \dots d$, be the different embeddings of K into the reals. The map $\varphi = (\sigma_1, \dots, \sigma_d)^t : K \to \mathbb{R}^d$ is called a *geometric embedding.* It is well known that if Λ is a lattice in K, then $\varphi(\Lambda)$ is a lattice in $\mathbb{R}^d$. The ring of integers in K is denoted by $\mathcal{O}_K$ and the group of units of this ring is denoted by $\mathcal{O}_K^*$. The logarithmic embedding of $\mathcal{O}_K^*$ in $\mathfrak{a}$ (see (2.1)) is given by $\omega \mapsto (\log|\sigma_1(\omega)|, \dots, \log|\sigma_d(\omega)|)$. We shall denote the image of $\mathcal{O}_K^*$ by Ω_K. Dirichlet's unit theorem implies that Ω_K is a lattice in $\mathfrak{a}$.

Theorem 3.1. *Let $x_0 \in X_d$. Ax_0 is compact if and only if there exists $a \in A$ such that ax_0 is (up to multiplication by a normalizing scalar) the geometric embedding of a lattice in a totally real number field, K, of degree d. Moreover there exists some finite index subgroup $\Omega < \Omega_K$ such that $\Omega = \log(Stab_A(x_0))$.*

As a corollary we get a classification of the compact A-orbits in Y_d. The proof is left to the reader.

Corollary 3.2. *A grid $y \in Y_d$ has a compact A-orbit, if and only if it is rational and $\pi(y) \in X_d$ has a compact orbit. In this case, $Stab_A(y)$ is of finite index in $Stab_A(\pi(y))$.*

The following corollary of theorem 3.1 is one of the places in which higher rank is reflected.

Corollary 3.3. *Let $d \geq 3$ and let $y \in Y_d$ be a grid with a compact A-orbit. Denote $A_0 = Stab_A(y)$. Then, for any root $\alpha \in \Phi$, the set $\{\alpha(a) : a \in A_0\}$ is dense in the reals.*

Proof. Let K be the totally real number field of degree d arising from theorem 3.1 and corollary 3.2 and let $\alpha \in \Phi$ be a root. By corollary 3.2, $\log(A_0)$ is of finite index in Ω_K. It follows that it is enough to justify why $\alpha(\Omega_K)$ is dense in the reals. As Ω_K is a lattice in $\mathfrak{a}$, this is equivalent to $\Omega_K \cap ker(\alpha)$ not being a lattice in $ker(\alpha)$. If α corresponds to a pair $i \neq j$ (see (2.2)), then if $\Omega_K \cap ker(\alpha) < ker(\alpha)$ is a lattice, then there is a subfield of K (the field $\{\theta \in K : \sigma_i(\theta) = \sigma_j(\theta)\}$) with a group of units containing a copy of $\mathbb{Z}^{d-2}$. The degree of this subfield is at most $d/2$ and so by Dirichlet's unit theorem the degree of the group of units in this subfield is at most $d/2 - 1$. This means that $d/2 - 1 \geq d - 2$ which is equivalent to $d \leq 2$ a contradiction. If α corresponds to k, then the situation is even simpler as $\Omega_K \cap ker(\alpha) = \{0\}$. □

4. Dynamics and *GDP* lattices

4.1. **Inheritance.** The reason that the action of A on X_d, Y_d is of importance to us is the invariance of the product set, namely $\forall a \in A, y \in Y_d$, $P(y) = P(ay)$.

Definition 4.1.

(1) A grid $y \in Y_d$ is called DP (dense products) if $\overline{P(y)} = \mathbb{R}$.

(2) *A lattice $x \in X_d$ is called GDP if any grid $y \in \pi^{-1}(x)$ is DP.*

The proofs of the next useful lemma and its corollary are left to the reader.

Lemma 4.2 (Inheritance)**.** *If $y, y_0 \in Y_d$ are such that $y_0 \in \overline{Ay}$, then $\overline{P(y_0)} \subset \overline{P(y)}$.*

Corollary 4.3.

(1) *If $y, y_0 \in Y_d$ are such that $y_0 \in \overline{Ay}$ and y_0 is DP, then y is DP too.*
(2) *If $x, x_0 \in X_d$ are such that $x_0 \in \overline{Ax}$ and x_0 is GDP, then x is GDP too.*

Lemma 4.4.

(1) *If $y \in Y_d$ is such that $\overline{Ay} \supset \pi^{-1}(x_0)$ for some $x_0 \in X_d$ then y is DP.*
(2) *If $y \in Y_d$ is such that there exists $y_0 \in Y_d$ and a root group $\{u_{ij}(t)\}_{t\in\mathbb{R}} < G_0$ such that $\overline{Ay} \supset \{u_{ij}(t)y_0 : t \in I\}$, where $I \subset \mathbb{R}$ is a ray, then y is DP.*

Proof. To see (1) note that from the inheritance lemma it follows that $\forall v \in \mathbb{R}^d$ $P(x_0+v) \subset \overline{P(y)}$. Clearly $\cup_{v\in\mathbb{R}^d} P(x_0 + v) = \mathbb{R}$. To see (2) note that it follows from [R] theorem B, that

$$\{u_{ij}(t)y_0 : t \in \mathbb{R}\} \subset \overline{\{u_{ij}(t)y_0 : t \in I\}}.$$

Let $w \in y_0$ be a vector all of whose coordinates are nonzero. By the inheritance lemma

$$\overline{P(y)} \supset \{N(u_{ij}(t)w) : t \in \mathbb{R}\} = \left\{N(w)\left(\frac{w_j}{w_i}t + 1\right) : t \in \mathbb{R}\right\} = \mathbb{R}.$$

□

4.2. **Existence of GDP lattices for $d \geq 3$.** The proof of the following theorem is based on the ideas presented in [LW].

Theorem 4.5. *If $x, x_0 \in X_d$ $(d \geq 3)$, Ax_0 is compact and $x_0 \in \overline{Ax} \setminus Ax$, then x is GDP.*

Proof. Let $y \in \pi^{-1}(x)$. Consider $F = \overline{Ay}$ and $F_0 = F \cap \pi^{-1}(x_0)$. Note that from the compactness of the fibers of π and the assumptions of the theorem it follows that $F_0 \neq \emptyset$. In [Sh] (lemma 4.8) it is shown that any irrational grid $y_0 \in \pi^{-1}(x_0)$, satisfies $\overline{Ay_0} \supset \pi^{-1}(x_0)$. Hence by lemma 4.4 (1), y_0 is DP. Hence, if F_0 contains an irrational grid then y is DP by corollary 4.3 (1).
Assume then that F_0 contains only rational grids and let $y_0 \in F_0$ (this could happen for example if y is a rational grid). By corollary 3.2, Ay_0 is compact. Denote $A_0 = Stab_A(y_0)$. Choose a regular element $b \in A_0$. Let $U^- = U^-(b), U^+ = U^+(b)$, be the corresponding stable and unstable horospherical subgroups of G. Any point which is close enough to y_0 in Y_d, has a unique representation of the form $au^+u^-y_0$, where $a \in A, u^+ \in U^+$ and $u^- \in U^-$ are in corresponding neighborhoods of the identity. Choose a sequence $y_n \to y_0$ from the orbit Ay. We may assume that

$$y_n = a_n u_n^+ u_n^- y_0 \in F \tag{4.1}$$

where $a_n, u_n^+, u_n^- \to e$. We may further assume that $a_n = e$ for all n, for if not, replace y_n by $a_n^{-1}y_n$. The fact that y_0 is not in Ay implies that the pairs (u_n^+, u_n^-) are nontrivial for

any n. Our first goal is to show:
Claim 1: *There exist a point in F of the form uy_0, where $u \neq e$ is in U^+ or U^-.*
If there exists an n with one of u_n^+ or u_n^-, being trivial, then the claim follows. If not, we denote for any n by k_n, the least integer such that the maximum of the absolute values of the entries of $b^{k_n}u_n^+b^{-k_n}$ is greater than 1. It then follows that this absolute value lies in some interval of the form $[1, M]$ (where M only depends on the choice of b). Since $u_n^+ \to e$ we must have $k_n \to \infty$. It is easy to see that the convergence $b^{k_n}u^-b^{-k_n} \to e$, for $u^- \in U^-$, is uniform on compact subsets of U^-. Hence in particular, $b^{k_n}u_n^-b^{-k_n} \to e$. Thus after going to a subsequence and abusing notation, we may assume that $b^{k_n}u_n^+b^{-k_n} \to u$, where $e \neq u \in U^+$. Hence

$$\lim b^{k_n}y_n = \lim b^{k_n}u_n^+b^{-k_n}b^{k_n}u_n^-b^{-k_n}y_0 = uy_0 \in F$$

and claim 1 follows.
Claim 2: *There exist a root $\alpha \in \Phi$ and $t_0 \neq 0$ such that $u_\alpha(t_0)y_0 \in F$.*
Let u be as in claim 1. We denote for $g \in G$,

$$\Phi_g = \{\alpha \in \Phi : \text{ the entry corresponding to } \alpha \text{ in } g \text{ is nonzero}\}.$$

If Φ_u contains only one root, claim 2 follows. If not, there exists a one parameter semigroup $\{a_t\}_{t\geq 0} < A$ such that Φ_u is the union of two non empty disjoint sets, Φ_u^-, Φ_u^0, such that for $\alpha \in \Phi_u^-$, $\alpha(a_1) < 0$, while for $\alpha \in \Phi_u^0$, $\alpha(a_1) = 0$ (see [LW] step 4.5 for details). It follows that for any sequence $t_n \to \infty$, $a_{t_n}ua_{t_n}^{-1} \to u'$, where $\Phi_{u'} = \Phi_u^0$, which is strictly smaller then Φ_u. Since $Ay_0 \simeq A/A_0$ is a $(d-1)$-torus, we can always find a sequence $t_n \to \infty$, such that $a_{t_n}y_0 \to y_0$. Thus

$$\lim a_{t_n}uy_0 = \lim a_{t_n}ua_{t_n}^{-1}a_{t_n}y_0 = u'y_0 \in F.$$

Repeating this process a finite number of times, we end up with a root α and some nonzero real number t_0, such that $u_\alpha(t_0)y_0 \in F$ and claim 2 follows.
Claim 3: *There exists a ray $I \subset \mathbb{R}$ such that $\{u_\alpha(t)y_0 : t \in I\} \subset F$.*
By corollary 3.3, we have that $\{\alpha(a) : a \in A_0\}$ is dense in $\mathbb{R}$. It follows that $I = \{e^{\alpha(a)}t_0 : a \in A_0\}$ is a ray. We have

$$\{au_\alpha(t_0)y_0 : a \in A_0\} = \left\{au_\alpha(t_0)a^{-1}y_0 : a \in A_0\right\} = \left\{u_\alpha(e^{\alpha(a)}t_0)y_0 : a \in A_0\right\} \subset F.$$

Claim 3 now follows from the fact that F is closed.
Note that from our assumption that F_0 contain only rational grids, it follow that the root group $u_\alpha(t)$ is contained in G_0. It now follows from lemma 4.4 (2) that y is DP and the theorem follows. □

Corollary 4.6. *For $d \geq 3$, any lattice with a dense A orbit is GDP.*

Proof. This is a consequence of theorem 4.5, and corollary 4.3(2). □

The following lemma is well known. We give the outline of a proof.

Lemma 4.7. *For any $d \geq 2$ and almost any $v \in \mathbb{R}^{d-1}$ (with respect to Lebesgue measure) $\overline{Ax_v} = \overline{Az_v} = X_d$.*

Proof. Let us consider lattices of the form x_v for example. Denote

$$a_t = diag(e^t, \ldots, e^t, e^{(1-d)t}).$$

Note that for any positive t the unstable horospherical subgroup of a_t is (recall the notation of §§ 1.4) $U^+(a_t) = \left\{h_v : v \in \mathbb{R}^{d-1}\right\}$. For any point $x \in X_d$ there exists neighborhoods W_x^+, W_x^-, W_x^0 of the identity elements in the groups $U^+(a_t), U^-(a_t)$ and the centralizer of a_t, such that the map $W_x^0 \times W_x^- \times W_x^+ \to X_d$ given by $(c, g, h_v) \mapsto cgh_vx$ is a diffeomorphism with a neighborhood, W_x, of x in X_d. Note that if $x_i = c_ig_ih_vx, i = 1, 2$ are two points in W_x having the same U^+ coordinate, then the trajectory $\{a_tx_1\}_{t\geq 0}$ is dense in X_d if and only if $\{a_tx_2\}_{t\geq 0}$ is dense too. As the action of a_t on X_d is ergodic we know that for almost any $x' \in W_x$, $\{a_tx'\}_{t\geq 0}$ is dense in X_d and from analyzing the structure of the Haar measure on X_d restricted to W_x we conclude that for almost any v in the neighborhood of zero in $\mathbb{R}^{d-1}$ corresponding to W_x^+, $\{a_th_vx\}_{t\geq 0}$ is dense in X_d. We abuse notation and think of W_x^+ as contained in $\mathbb{R}^{d-1}$.
To finish the argument we find a countable collection $v_i \in \mathbb{R}^{d-1}$ such that the neighborhoods $W_{x_{v_i}}$ satisfy $\mathbb{R}^{d-1} = \cup_i \left(v_i + W_{x_{v_i}}\right)$ and note that $W_{x_{v_i}}x_{v_i} = \left\{x_w : w \in v_i + W_{x_{v_i}}\right\}$. □

Corollary 4.8. *Let $d \geq 3$*

(1) Almost any lattice $x \in X_d$ (with respect to Haar measure) is GDP.
(2) For almost any $v \in \mathbb{R}^{d-1}$ (with respect to Lebesgue measure), both $x_v, z_v \in X_d$ are GDP.
(3) If $v \in \mathbb{R}^{d-1}$ is such that x_v (resp z_v) is GDP, then (1.7) *(resp* (1.8)*) is satisfied.*

Proof. (1) follows from the ergodicity of the A action on X_d which in particular means that almost any point has a dense orbit and corollary 4.6. (2) follows from lemma 4.7 and corollary 4.6. (3) is left to be verified by the reader. □

5. A density result

Let $x_0 \in X_3$ be a point with a compact A-orbit. We shall use the following facts: It follows from Lemma 4.1 of [LW], that the orbit of x_0 under any root group $u_{ij}(t)$, is dense in X_3, moreover Theorem B of [R], implies that in fact $\{u_{ij}(t)x_0\}_{t\in I}$ is dense in X_3, for any ray $I \subset \mathbb{R}$. It follows from corollary 1.4 in [LW], that if $b \in G_0$ is lower or upper triangular but not diagonal, then $\overline{Abx_0} = X_3$. A more careful look yields the following theorem. The author is indebted to Elon Lindenstrauss, for valuable ideas appearing in the proof.

Theorem 5.1. *Let $x_0 \in X_3$ be a lattice with a compact A-orbit. If $p = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & \delta \\ \eta & \tau & \mu \end{pmatrix} \in G_0$ is such that both $\tau, \mu \neq 0$, then $\overline{Apx_0} = X_3$.*

Proof. A straightforward computation shows

$$p=\begin{pmatrix}1&0&0\\0&1&\frac{\delta}{\mu}\\0&0&1\end{pmatrix}\begin{pmatrix}\alpha&0&0\\\beta-\frac{\delta\eta}{\mu}&\gamma-\frac{\delta\tau}{\mu}&0\\\eta&0&\mu\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&\frac{\tau}{\mu}&1\end{pmatrix}=u_{23}(t_0)b_1b_2, \tag{5.1}$$

where we denoted $t_0=\frac{\delta}{\mu}$ and the matrices appearing in the middle of (5.1) by $u_{23}\left(t_0\right), b_1$ and b_2 according to appearance. Note that the matrix $b=b_1b_2$ is nondiagonal as $\tau\neq 0$, hence by the preceding discussion, if we denote $x_1=bx_0$, then x_1 has a dense A-orbit. Hence, it is enough to show that x_1 belongs to the orbit closure of $px_0=u_{23}\left(t_0\right)x_1$. This will follow from the existence of a recurrence sequence $a_n\in A$ for x_1 (i.e. a sequence such that $a_nx_1\to x_1$) which in addition satisfies $a_nu_{23}\left(t_0\right)a_n^{-1}\to e$, for then

$$\lim a_npx_0=\lim a_nu_{23}(t_0)a_n^{-1}a_nx_1=x_1. \tag{5.2}$$

A sequence a_n satisfies $a_nu_{23}(t_0)a_n^{-1}\to e$, if and only if $t_2^{(n)}-t_3^{(n)}\to-\infty$, where $\mathbf{t}^{(n)}=\log(a_n)$. Thus it is enough to show that for any $m>0$, there exists a recurrence sequence for x_1 in $A_m=\exp\left(R_m\right)$, where $R_m=\{\mathbf{t}\in\mathfrak{a}:t_2-t_3\leq -m\}$ is a half plane. Choose $m>0$. We shall show that in fact A_mx_1 is dense in X_3. Denote

$$a_t=diag\left(e^{2t},e^{-t},e^{-t}\right)\text{ and }a'=diag\left(e^{-m},1,e^{m}\right). \tag{5.3}$$

The line $\{a'a_t\}_{t\in\mathbb{R}}$ lies on the boundary of A_m. As $b=b_1b_2$, we write (emphasizing the desired partition into products)

$$A_mx_1\supset a'a_tx_1=a'a_tbx_0=\left(a'a_tb_1\left(a'a_t\right)^{-1}\right)\cdot\left(\left(a'a_t\right)b_2\left(a'a_t\right)^{-1}\right)\cdot\left(\left(a'a_t\right)x_0\right). \tag{5.4}$$

We observe that for any sequence $t_n\to\infty$, $a'a_{t_n}b_1\left(a'a_{t_n}\right)^{-1}$ converges to the diagonal matrix $a''=diag\left(\alpha,\gamma-\frac{\delta\tau}{\mu},\mu\right)$, while at the same time $\left(a'a_{t_n}\right)b_2\left(a'a_{t_n}\right)^{-1}$ converges to $u_{32}\left(s_0\right)$, where $s_0=e^m\frac{\tau}{\mu}\neq 0$. Furthermore, because $Ax_0\simeq A/Stab_A(x_0)$, and because the line a_t is irrational with respect to the lattice $Stab_A(x_0)$ (by theorem 3.1), any trajectory of $\{a_t\}_{t\geq 0}$ in Ax_0 is dense there. In particular any point in Ax_0 is a limit point of some sequence $\left(a'a_{t_n}\right)x_0$, for some sequence $t_n\to\infty$. It follows now from (5.4), that

$$\overline{A_mx_1}\supset a''u_{32}\left(s_0\right)Ax_0=u_{32}(s_1)Ax_0, \tag{5.5}$$

for a suitable choice of $s_1\neq 0$. As A_m is closed under multiplication, $\overline{A_mx_1}$ closed under the action of A_m. In particular, it follows from (5.5), that for any $a\in A_m$, $au_{32}(s_1)a^{-1}x_0\in\overline{A_mx_1}$, i.e. $u_{32}(s)x_0\in\overline{A_mx_1}$, where s ranges over the set $\{e^ts_1:t\geq m\}$, which is a ray. The discussion preceding this proof now implies the density of A_mx_1 and in particular that $x_1\in\overline{A_mx_1}$ as desired. □

Corollary 5.2. *Let K be a totally real cubic number field and let $1,\alpha,\beta$ be a basis of K over $\mathbb{Q}$. Denote $v=(\alpha,\beta)^t\in\mathbb{R}^2$. Then the lattices x_v,z_v, have dense A orbits in X_3 and in particular they are GDP by corollary 4.6.*

Proof. Let us denote $\alpha = \alpha_1, \beta = \beta_1$ and let $\alpha_i, \beta_i, i = 2, 3$, be the other two embeddings of α_1, β_1 into the reals. Denote $g_0 = c \begin{pmatrix} 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_3 & \beta_3 \end{pmatrix}$, where c is chosen so that $\det(g_0) = 1$. Then, $\bar{g}_0 \in X_3$ has a compact A-orbit by theorem 3.1. It is easy to see that there exists a unique matrix $p \in G_0$ as in lemma 5.1, such that (recall the notation of §§ 1.4)

$$pg_0 = g_v. \tag{5.6}$$

The reader can easily check that the relevant entries of p must be nonzero. We apply lemma 5.1, and conclude that $\bar{g}_v = z_v$ has a dense A-orbit in X_3. In order to see that x_v has a dense orbit, we note that the involution $g \mapsto (g^t)^{-1} = g^*$ of G_0 , descends to a diffeomorphism of X_3. We denote this map by $\bar{g} \mapsto \bar{g}^* = \overline{g^*}$. This is the well known map which sends a lattice to its dual. Since the group A is invariant under this involution, we have that for any lattice x, $\left(\overline{Ax}\right)^* = \overline{Ax^*}$. In particular, x has a dense orbit if and only if x^* has. In a similar way to what we have already shown, one can show that the lattice spanned by the columns of $g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & -\beta & 1 \end{pmatrix}$, has a dense A-orbit, in X_3. As $g_1^* = h_v$, it follows that $\bar{h}_v = x_v$ has a dense orbit too, as desired. $\square$

6. Irregular A orbits

In this section we use the existence of lattices $x \in X_d$ $(d \geq 3)$ for which $\mu(x) = 0$ (theorem 1.2), and theorem 1.3(2) to give examples of lattices in X_3 having irregular A orbit closures of a new type. This serves as a counterexample to conjecture 1.1 in [Ma]. Our example proceeds the recent counterexample, F .Maucourant gave to this conjecture in [Mau]. Our example is different in nature from Maucourant's example. We use a maximal split torus whilst in [Mau] the acting group does not "separate roots" which seem to be the reason for the abnormality. It still seems plausible that a slightly different version of that conjecture will be true.

Proof of theorem 1.3(2). We first note that if $x_1, x_2 \in X_d$ are commensurable lattices (that is their intersection is of finite index in each), then $\mu(x_1) = 0$ if and only if $\mu(x_2) = 0$. Let $v \in \mathbb{R}^{d-1}$ satisfy $\dim_{\mathbb{Q}} \mathit{span}\{1, v_1 \ldots, v_{d-1}\} \leq 2$. Then, there exists $\alpha \in \mathbb{R}$ and rationals $p_i, q_i, i = 1 \ldots d-1$ such that $v_i = q_i\alpha + p_i$. Denote $v' = (q_1\alpha, \ldots, q_{d-1}\alpha)^t$. It follows that x_v, z_v are commensurable to $x_{v'}, z_{v'}$ respectively.

Claim 1: $\mu(x_{v'}) > 0$. Working with the definition of μ we see that it is enough to argue the existence of $d-1$ real numbers γ_i for which

$$N\left(x_{v'} + (-\gamma_1, \ldots, -\gamma_{d-1}, 1/2)^t\right) = \inf_{n \in \mathbb{Z}} |n + 1/2| \prod_1^{d-1} \langle nq_i\alpha - \gamma_i \rangle > 0. \tag{6.1}$$

From Davenport's result described in §§ 1.2 it follows that there exists $\gamma_i \in \mathbb{R}$ such that for each i, $\inf_{n\in\mathbb{Z}} |n + 1/2| \langle nq_i\alpha - \gamma_i \rangle > 0$. Moreover, if we denote by m, a common denominator for the q_i's , then by [Ca2] (theorem 1), we can choose the γ_i's such that

for any $i \neq j$, $\frac{\gamma_i}{q_i} - \frac{\gamma_j}{q_j} \notin \frac{1}{m}\mathbb{Z}$. Denote for $r, s \in \mathbb{R}$ by $\langle r - s\rangle_m$ the distance modulo $\frac{1}{m}\mathbb{Z}$ from r to s. Denote $\rho = \min_{i\neq j}\langle \frac{\gamma_i}{q_i} - \frac{\gamma_j}{q_j}\rangle_m$. Note that for $\epsilon > 0$, $\langle nq_i\alpha - \gamma_i\rangle < \epsilon \Rightarrow \langle n\alpha - \frac{\gamma_i}{q_i}\rangle_m < \frac{\epsilon}{|q_i|}$. Hence if $\max_i \frac{\epsilon}{|q_i|} < \rho/2$, then $\langle nq_i\alpha - \gamma_i\rangle < \epsilon$ for at most one index i. Let $\epsilon > 0$ be such. Assume that the left hand side of (6.1) is smaller than $\epsilon^{d-1}/2$. Then for some k, $\langle nq_k\alpha - \gamma_k\rangle < \epsilon$ which implies that the left hand side of (6.1) is $> \epsilon^{d-2}\inf_{n\in\mathbb{Z}} |n + 1/2| \langle nq_k\alpha - \gamma_k\rangle > 0$ as desired.
Claim 2 $\mu(z_{v'}) > 0$. We use the notation as in Claim 1. From Davenport's result, we know that there exists $0 \neq \gamma \in \mathbb{R}$ such that $\inf_{k\in\mathbb{Z}} |k + 1/2| \langle k(\alpha/m) - \gamma\rangle > 0$. The reader will easily argue the existence of a constant $c > 0$ for which

$$\forall \vec{n} \in \mathbb{Z}^{d-1} \setminus \{0\}, \quad \prod_1^{d-1} |n_i + 1/2| \geq c\,|m\vec{q}\cdot\vec{n} + 1/2|.$$

Working with the definition of μ we see that

$$\begin{aligned} \mu(z_{v'}) &\geq N\left(z_{v'} + (\gamma, 1/2, \ldots, 1/2)^t\right) = \inf_{\vec{n}\in\mathbb{Z}^{d-1}} \prod |n_i + 1/2| \left\langle (m\vec{q}\cdot\vec{n})\,\alpha/m - \gamma\right\rangle \\ &\geq \min\left\{\inf_{\vec{n}\cdot\vec{q}\neq 0} c\,|m\vec{q}\cdot\vec{n} + 1/2| \left\langle (m\vec{q}\cdot\vec{n})\,\alpha/m - \gamma\right\rangle; \tfrac{|\gamma|}{2^{d-1}}\right\} > 0. \end{aligned}$$

□

Observe that if $x_n \to x$ in X_d, then $\limsup \mu(x_n) \leq \mu(x)$. It follows that for $x_0, x \in X_d$

$$x_0 \in \overline{Ax} \Rightarrow \mu(x_0) \geq \mu(x). \tag{6.2}$$

By theorem 1.2, if $x \in X_d$, $(d \geq 3)$ and $\mu(x) > 0$, then Ax is not dense.
The following statement is a special case of Conjecture 1.1 of [Ma]:

Conjecture 6.1 (Special case of conjecture 1.1 in [Ma])**.** *For $x \in X_3$, one of the following three options occurs:*

1. *Ax is dense.*
2. *Ax is closed.*
3. *Ax is contained in a closed orbit Hx of an intermediate group $A < H < G$, where H could be one of the following three subgroups of G_0:*

$$H_1 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, H_2 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, H_3 = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}.$$

For $t \in \mathbb{R}$ denote $v_t = (t, t)^t \in \mathbb{R}^2$. We denote the one parameter group h_{v_t} (recall the notation of §§ 1.4) simply by h_t and the lattice x_{v_t} by x_t.

Theorem 6.2. *There exists $t \in \mathbb{R}$ such that $x_t \in X_3$ violates conjecture 6.1.*

Proof. By theorem 1.3(2), $\mu(x_t) > 0$ hence by theorem 1.2 and (6.2), possibility (1) is ruled out. To rule out possibilities (2) and (3), we note that if H is either one of the groups A, H_1, H_2, H_3, then for any $g \in G_0$, $H\bar{g}$ is closed in X_3 if and only if $g^{-1}Hg$ is defined over $\mathbb{Q}$. Assume to get a contradiction that for any $t \in \mathbb{R}$, for H equals one of the above, the group $h_t^{-1}Hh_t$ is defined over $\mathbb{Q}$. Two elements $g_1, g_2 \in G_0$ conjugate H to the

same group if and only if $g_1g_2^{-1}$ normalizes H. All the above groups are of finite index in their normalizers in G_0 and so there exists some k such that whenever g normalizes H, then $g^k \in H$. As there are only countably many $\mathbb{Q}$-groups in G_0, there must exist some $t \neq s$ such that $(h_th_s^{-1})^k = h_{k(t-s)} \in H$ which of course never happens. $\square$

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